JEE Main 2023 (Memory based)

29 January 2023 - Shift 2

Answer & Solutions

MATHEMATICS

1. The 3 digit numbers which are divisible by either 3 or 4 but not divisible by 48:

A. 414
B. 420
C. 429
D. 432

Answer (D)
Solution:

No's divisible by $3 = 300$

No's divisible by $4 = 225$

No's divisible by $12 = 75$

No's divisible by $48 = 18$

Total numbers $= 300 + 225 - 75 - 18$

$= 432$

2. The letters of word GHOTU is arranged alphabetically as in a dictionary. The rank of the word TOUGH is:

A. 84
B. 79
C. 74
D. 89

Answer (D)

Solution:

Number of words starting with

G _ _ _ _ $= 4! = 24$

H _ _ _ _ $= 4! = 24$

O _ _ _ _ $= 4! = 24$

T G _ _ _ $= 3! = 6$

T H _ _ _ $= 3! = 6$

T O G _ _ $= 2! = 2$

T O H _ _ $= 2! = 2$

T O U G H $= 1$

$\therefore$ Rank of word TOUGH is $= 24 \times 3 + 6 \times 2 + 2 \times 2 + 1 = 89$

3. $\int_{\frac{1}{2}}^{2} \frac{\tan^{-1} x}{x} dx$ equals:

A. $\frac{\pi}{2} \ln 2$
B. $\frac{\pi}{4} \ln 2$
C. $\pi \ln 2$
D. $\ln 2$

Answer (A)

Solution:

$I = \int_{\frac{1}{2}}^{2} \frac{\tan^{-1} x}{x} dx \quad \cdots (1)$

Put $x = \frac{1}{t}$

$dx = -\frac{1}{t^2} dt$

$I = \int_{2}^{\frac{1}{2}} \frac{\tan^{-1}\left(\frac{1}{t}\right)}{\frac{1}{t}} \left(-\frac{1}{t^2}\right) dt$

$I = \int_{\frac{1}{2}}^{2} \frac{\tan^{-1}\left(\frac{1}{t}\right)}{t} dt$

$I = \int_{\frac{1}{2}}^{2} \frac{\cot^{-1} t}{t} dt \quad \cdots (2)$

By adding eq. (1) and eq. (2)

$2I = \int_{\frac{1}{2}}^{2} \frac{\pi}{2} \frac{dt}{t} \quad \cdots \left(\because \tan^{-1} t + \cot^{-1} t = \frac{\pi}{2}\right)$

$\Rightarrow I = \frac{\pi}{2} \ln 2$

4. Shortest distance between lines: $\frac{x-1}{2} = \frac{2y-2}{3} = \frac{z-3}{1}$ and $\frac{x-2}{3} = \frac{y-1}{2} = \frac{z+2}{4}$ is:

A. $\frac{13}{\sqrt{165}}$

B. $\frac{15}{\sqrt{165}}$

C. $\frac{18}{\sqrt{165}}$

D. $\frac{19}{\sqrt{165}}$

Answer (A)

Solution:

For line $\frac{x-1}{2} = \frac{2y-2}{3} = \frac{z-3}{1}$

$\overrightarrow{a_1} = \hat{\imath} + \hat{\jmath} + 3\hat{k}$

$\overrightarrow{b_1} = 2\hat{\imath} + \frac{3}{2}\hat{\jmath} + \hat{k}$

For line $\frac{x-2}{3} = \frac{y-1}{2} = \frac{z+2}{4}$

$\overrightarrow{a_2} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}$

$\overrightarrow{b_2} = 3\hat{\imath} + 2\hat{\jmath} + 4\hat{k}$

$$\overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & \frac{3}{2} & 1 \\ 3 & 2 & 4 \end{vmatrix} = 4\hat{\imath} - 5\hat{\jmath} - \frac{\hat{k}}{2}$$

$$(\overrightarrow{a_2} - \overrightarrow{a_1}) \cdot (\overrightarrow{b_1} \times \overrightarrow{b_2}) = (\hat{\imath} - 5\hat{k}) \cdot \left(4\hat{\imath} - 5\hat{\jmath} - \frac{\hat{k}}{2}\right)$$

$$= 4 + \frac{5}{2} = \frac{13}{2}$$

Shortest distance $= \left|\frac{[\overrightarrow{a_2} - \overrightarrow{a_1} \quad \overrightarrow{b_1} \quad \overrightarrow{b_2}]}{|\overrightarrow{b_1} \times \overrightarrow{b_2}|}\right| = \left|\frac{(\overrightarrow{a_2} - \overrightarrow{a_1}) \cdot (\overrightarrow{b_1} \times \overrightarrow{b_2})}{|\overrightarrow{b_1} \times \overrightarrow{b_2}|}\right|$

$$= \frac{\frac{13}{2}}{\sqrt{16+25+\frac{1}{4}}} = \frac{13}{\sqrt{165}}$$

5. $R = \{(a, b) : 2a + 3b$ is divisible by 5 and $a, b \in N\}$ is:

A. Transitive but not symmetric
B. Equivalence Relation
C. Symmetric but not Transitive
D. Not Equivalence

Answer (B)

Solution:

$f(a, b) = 2a + 3b$
For reflexive
$f(a, a) = 2a + 3a = 5a$ i.e, divisible by 5
$\Rightarrow (a, a) \in R$
For symmetric
$$f(b, a) = 2b + 3a = \underbrace{5a + 5b}_{\text{Divisible by } 5} - \underbrace{(2a + 3b)}_{\text{Divisible by } 5}$$
$f(b, a)$ is divisible by $5 \Rightarrow (b, a) \in R$
For transitive
$f(a, b) = 2a + 3b$ is divisible by 5
$f(b, c) = 2b + 3c$ is divisible by 5
$\Rightarrow \; 2a + 5b + 3c$ is divisible by 5
So, $2a + 3c$ is divisible by 5
$\Rightarrow (a, c) \in R$

6. $(\sim A) \vee B$ is equivalent to:

A. $A \rightarrow B$
B. $A \leftrightarrow B$
C. $\sim A \wedge B$
D. $B \rightarrow A$

Answer (A)

Solution:

Making truth table,

$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{\sim A}$	$(\boldsymbol{\sim A}) \vee \boldsymbol{B}$	$\boldsymbol{A \rightarrow B}$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The truth table clearly shows $(\sim A) \vee B \equiv A \rightarrow B$

7. The value of $\int_{\frac{1}{2}}^{2}\left(\frac{t^4+1}{t^6+1}\right)dt$:

A. $\tan^{-1}2+\tan^{-1}8+\frac{2\pi}{3}$

B. $2\tan^{-1}2+\frac{2}{3}\tan^{-1}8-\frac{2\pi}{3}$

C. $2\tan^{-1}2+\frac{2}{3}\tan^{-1}8+\frac{2\pi}{3}$

D. $2\tan^{-1}2-\frac{2}{3}\tan^{-1}8+\frac{2\pi}{3}$

Answer (B)

Solution:

$$\int_{\frac{1}{2}}^{2}\left(\frac{t^4+1}{t^6+1}\right)dt=\int_{\frac{1}{2}}^{2}\frac{(t^4+1)(t^2+1)}{(t^6+1)(t^2+1)}dt$$
$$=\int_{\frac{1}{2}}^{2}\frac{t^6+1+t^2(t^2+1)}{(t^6+1)(t^2+1)}dt$$
$$=\int_{\frac{1}{2}}^{2}\frac{dt}{(t^2+1)}+\frac{1}{3}\int_{\frac{1}{2}}^{2}\frac{3t^2dt}{t^2+1}$$
$$=\tan^{-1}t\,\Big|_{\frac{1}{2}}^{2}+\frac{1}{3}\tan^{-1}t^3\,\Big|_{\frac{1}{2}}^{2}$$
$$=\left(\tan^{-1}2-\tan^{-1}\left(\frac{1}{2}\right)\right)+\frac{1}{3}\left(\tan^{-1}8-\tan^{-1}\left(\frac{1}{8}\right)\right)$$
$$=(\tan^{-1}(2)-\cot^{-1}(2)\quad)+\frac{1}{3}(\tan^{-1}(8)-\cot^{-1}(8))$$
$$=\left(\tan^{-1}2-\left(\frac{\pi}{2}-\tan^{-1}(2)\right)\right)+\frac{1}{3}\left(\tan^{-1}(8)-\left(\frac{\pi}{2}-\tan^{-1}(8)\right)\right)$$
$$=2\tan^{-1}2+\frac{2}{3}\tan^{-1}(8)-\frac{\pi}{2}-\frac{\pi}{6}$$
$$=2\tan^{-1}2+\frac{2}{3}\tan^{-1}8-\frac{2\pi}{3}$$

8. Area of region $|\cos x-\sin x|\le y\le\sin x$ for $x\in\left(0,\frac{\pi}{2}\right)$ is:

A. $\left(-1+2\sqrt{2}\right)$ sq. units

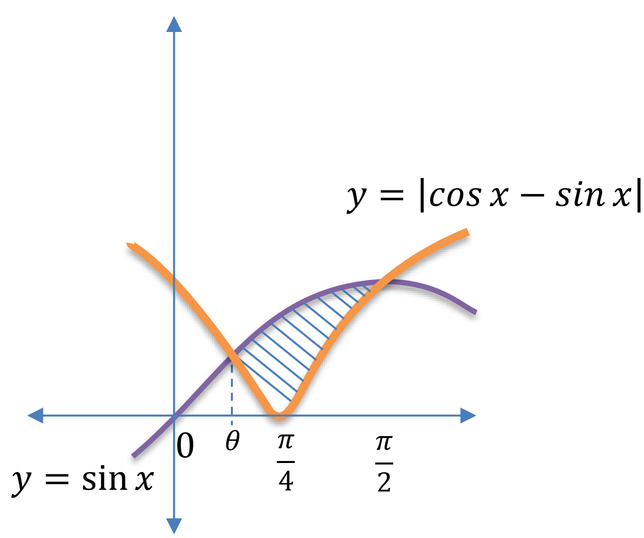
B. $\left(1-\frac{1}{\sqrt{2}}\right)$ sq. units

C. $\left(\sqrt{5}+1-2\sqrt{2}\right)$ sq. units

D. $\left(\sqrt{5}-\sqrt{2}\right)$ sq. units

Answer:(C)

Solution:



$$A=\int_{\theta}^{\frac{\pi}{2}}(\sin x-|\cos x-\sin x|)dx\text{ where }\theta=\tan^{-1}\frac{1}{2}$$
$$A=\int_{\theta}^{\frac{\pi}{4}}(\sin x-\cos x+\sin x)dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\sin x+\cos x-\sin x)dx$$
$$A=-2\cos x-\sin x|_{\theta}^{\frac{\pi}{4}}+\sin x|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$A=-\left(\sqrt{2}+\frac{1}{\sqrt{2}}-2\cos\theta-\sin\theta\right)\ +\left(1-\frac{1}{\sqrt{2}}\right)$$
$$A=-\sqrt{2}-\frac{1}{\sqrt{2}}+(2\cos\theta+\sin\theta)+\left(1-\frac{1}{\sqrt{2}}\right)$$
$$A=1-2\sqrt{2}+2.\frac{2}{\sqrt{5}}+\frac{1}{\sqrt{5}}\ (since\ \tan\theta=2)$$
$$A=\sqrt{5}+1-2\sqrt{2}$$

9. For solution of differential equation $x\ln x\frac{dy}{dx}+y=x^2\ln x$, $y(2)=2$, then $y(e)$ is equal to:

A. $1+\frac{e^2}{4}$

B. $1-\frac{e^2}{4}$

C. $\frac{e^2}{2}$

D. $1+\frac{e^2}{2}$

Answer (A)

Solution:

$x\ln x\frac{dy}{dx}+y=x^2\ln x$

$\frac{dy}{dx}+\frac{y}{x\ln x}=x$

I.F $=e^{\int\frac{1}{x\ln x}dx}=e^{\ln|\ln x|}=|\ln x|$

Solution of equation is,

$y\cdot(I.F)=\int x\cdot|\ln x|dx$

$y\cdot|\ln x|=|\ln x|\frac{x^2}{2}-\frac{x^2}{4}+C$

Put $x=2$

$\Rightarrow 2|\ln 2|=|\ln 2|\cdot 2-1+C$

$\Rightarrow C=1$

Put $x=e$

$y=\frac{e^2}{2}-\frac{e^2}{4}+1$

$y(e)=1+\frac{e^2}{4}$

10. Let $f(x)=x^2+2x+5$ and α,β be roots of $f\left(\frac{1}{t}\right)=0$, then $\alpha+\beta=$

A. $-\frac{2}{5}$

B. -2

C. $\frac{5}{2}$

D. $-\frac{5}{2}$

Answer (A)

Solution:

$f(x)=x^2+2x+5$

$f(t)=0$

$\Rightarrow \frac{1}{t^2}+\frac{2}{t}+5=0$

$\Rightarrow 5t^2+2t+1=0 \quad (t\neq 0)$

This equation has roots α and β

$\therefore \alpha+\beta=-\frac{2}{5}$

11. If the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+3}{1}$ and $\frac{x-11}{4}=\frac{y-9}{2}=\frac{z-4}{3}$ intersects at point P, then perpendicular distance of P from plane $3x+2y+6z=10$ is:

A. $\frac{2}{7}$
B. $\frac{3}{7}$
C. $\frac{4}{7}$
D. $\frac{5}{7}$

Answer (B)

Solution:

$L_1 \equiv \frac{x-1}{2}=\frac{y-2}{3}=\frac{z+3}{1}=\lambda$

$L_2 \equiv \frac{x-11}{4}=\frac{y-9}{2}=\frac{z-4}{3}=\mu$
$x=2\lambda+1=4\mu+11 \quad \cdots(1)$
$z=\lambda-3=3\mu+4 \quad \cdots(2)$
By solving eq.(1) and eq.(2)
We get $\lambda=1,\ \mu=-2$
∴ Point of intersection of the two lines
$x=3, y=5, z=-2$
$\Rightarrow P \equiv (3,5,-2)$
Distance from given plane $=\left|\frac{9+10-12-10}{\sqrt{9+4+36}}\right|=\frac{3}{7}$

12. If $\cos^2 2x-\sin^4 x-2\cos^2 x=\lambda$ has a solution $\forall\ x \in \mathbb{R}$, then the range of λ is:

A. $\left[-\frac{1}{2},\ 1\right]$
B. $\left[-\frac{4}{3},\ 0\right]$
C. $(0,\ 2)$
D. None of these

Answer (B)

Solution:
$\cos^2 2x-\sin^4 x-2\cos^2 x=\lambda$
$\Rightarrow (\cos^2 x-\sin^2 x)^2-\sin^4 x-2\cos^2 x=\lambda$
$\Rightarrow 3\cos^4 x-4\cos^2 x=\lambda$
$\Rightarrow 3\left(\left(\cos^2 x-\frac{2}{3}\right)^2-\frac{4}{9}\right)=\lambda$
$\Rightarrow \lambda_{\min}=-\frac{4}{3}\ \&\ \lambda_{\max}=0$
$\Rightarrow \lambda \in \left[-\frac{4}{3},\ 0\right]$

13. $\vec{a}=9\hat{\imath}+2\hat{k}, \vec{b}=\hat{\imath}+\hat{\jmath}+\hat{k}$ and $\vec{c}=7\hat{\imath}-3\hat{\jmath}+2\hat{k}$ are three given vectors. Let there be a $\vec{r}$ such that $(\vec{r}\times\vec{b})+(\vec{b}\times\vec{c})=0$ and $\vec{r}\cdot\vec{a}=0$ then $\vec{r}\cdot\vec{c}$ is ______.

A. $\frac{280}{11}$
B. 28
C. $\frac{279}{13}$
D. $\frac{290}{11}$

Answer (A)

Solution:

$(\vec{r}\times\vec{b})+(\vec{b}\times\vec{c})=0$

$(\vec{r}\times\vec{b})-(\vec{c}\times\vec{b})=0$

$(\vec{r}-\vec{c})\times\vec{b}=0$

$\Rightarrow \vec{r}=\vec{c}+\lambda\vec{b}$

$\vec{r}\cdot\vec{a}=0$ $\cdots$ (given)

$\vec{c}\cdot\vec{a}+\lambda\vec{b}\cdot\vec{a}=0$

$67+\lambda(11)=0$

$\lambda=-\frac{67}{11}$

$\vec{r}\cdot\vec{c}=(\vec{c}+\lambda\vec{b})\cdot\vec{c}$

$\vec{r}\cdot\vec{c}=|\vec{c}|^2+\lambda\vec{b}\cdot\vec{c}$

$=62-\frac{67}{11}(7-3+2)$

$=62-\frac{67}{11}(6)$

$\vec{r}\cdot\vec{c}=\frac{682-402}{11}=\frac{280}{11}$

14. For observation set x data obtained is $x_i=\{11,12,13,\dots,41\}$

For another observation set y data obtained is $y_i=\{61,62,63,\dots,91\}$

Then value of $\bar{x}+\bar{y}+\sigma^2$ where $\bar{x},\bar{y}$ are means of respective data set while σ^2 is the variance of combined data is :

A. 801
B. 754
C. 807
D. 774

Answer (C)

Solution:

$\bar{x}=\frac{\frac{31}{2}(11+41)}{31}=\frac{1}{2}\times 52=26$

$\bar{y}=\frac{\frac{31}{2}(61+91)}{31}=\frac{1}{2}\times 152=76$

$\sigma^2=\frac{\sum x_i^2+\sum y_i^2}{62}-\left(\frac{\sum x_i+\sum y_i}{62}\right)^2$

$\sigma^2=\frac{(11^2+12^2+13^2+\cdots 41^2)+(61^2+62^2+\cdots 91^2)}{62}-51^2$

$\sigma^2=\frac{\left(\frac{41\times42\times83}{6}-\frac{10\times11\times21}{6}\right)+\left(\frac{91\times92\times183}{6}-\frac{60\times61\times121}{6}\right)}{62}-(51)^2$

$\sigma^2=\frac{(41\times7\times83-11\times35)+(91\times46\times61-10\times61\times121)}{62}-(51)^2$

$\sigma^2=\frac{23436+181536}{62}-(51)^2$

$\sigma^2=3306-2601=705$

$\therefore \bar{x}+\bar{y}+\sigma^2=26+76+705=807$

15. If the curve represented by $y = \frac{(x-a)}{(x-3)(x-2)}$ passes through $(1, -3)$ then equation of normal at $(1, -3)$ to the curve is given by

A. $2x + 3y = -7$
B. $3x - 2y = 9$
C. $3x - 4y = 21$
D. $x - 4y = 13$

Answer (D)

Solution:

Curve $y = \frac{(x-a)}{(x-3)(x-2)}$ passes through $(1, -3)$

$\Rightarrow -3 = \frac{(1-a)}{(-2)(-1)}$

$\Rightarrow a = 7$

$f(x) = \frac{(x-7)}{(x-3)(x-2)}$

$f'(x) = \frac{(x-3)(x-2)-(x-7)(2x-5)}{((x-3)(x-2))^2}$

$f'(1) = \frac{2-18}{2^2} = -4$

Slope of normal $= \frac{-1}{-4} = \frac{1}{4}$

Equation of normal:

$y + 3 = \frac{1}{4}(x - 1)$

$\Rightarrow 4y + 12 = x - 1$

$\Rightarrow x - 4y = 13$

16. The number of four-digit numbers N such that $GCD(N, 54) = 2$ is ______.

Answer (3000)

Solution:

N should be divisible by 2 but not by 3.

$N =$ (number of numbers divisible by 2) $-$ (number of number divisible by 6)

$N = \frac{9000}{2} - \frac{9000}{6} = 3000$

17. If $f(1) + 2f(2) + 3f(3) + \cdots + nf(n) = n(n+1)f(n)$ and $f(1) = 1$, then $\frac{1}{f(2022)} + \frac{1}{f(2028)}$ is equal to ______.

Answer (4050)

Solution:

We have,

$f(1) + 2f(2) + 3f(3) + \cdots + nf(n) = n(n+1)f(n) \cdots (i)$

Replacing n by $n + 1$ in (i)

$f(1) + 2f(2) + 3f(3) + \cdots + nf(n) + (n+1)f(n+1) = (n+1)(n+2)f(n+1) \cdots (ii)$

Using (i) in (ii) we have:

$n(n+1)f(n) + (n+1)f(n+1) = (n+1)(n+2)f(n+1)$

$\Rightarrow f(n+1) = \left(\frac{n}{n+1}\right) f(n)$

$\because f(1) = 1$

$\Rightarrow f(2) = \frac{1}{2}$

$\Rightarrow f(3) = \frac{1}{3}$

$\dots$

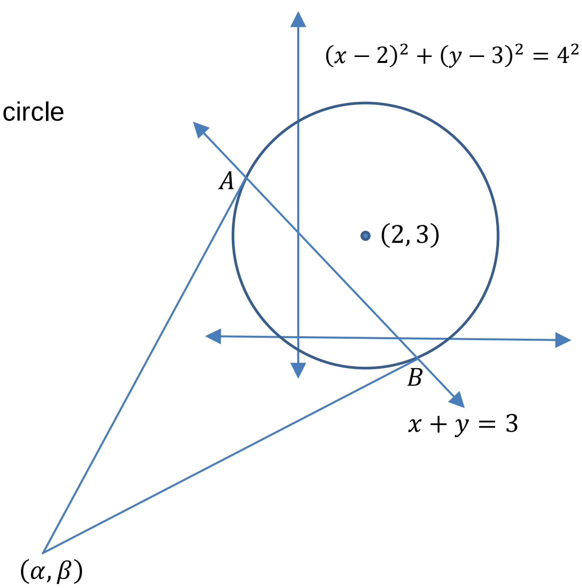
$\Rightarrow f(n) = \frac{1}{n}$

$$\frac{1}{f(2022)} + \frac{1}{f(2028)} = 2022 + 2028 = 4050$$

18. A line $x + y = 3$ cuts the circle having centre $(2,\ 3)$ and radius 4 at two points A and B. Tangents drawn at A and B intersect at $(\alpha,\ \beta)$. Then the value of $4\alpha - 7\beta$ is __________.

Answer (11)

Solution:



The given line $x + y = 3$ is the chord of contact of $(\alpha,\ \beta)$ w.r.t given circle

Circle Equation: $(x-2)^2 + (y-3)^2 = 4^2$

$\Rightarrow x^2 + y^2 - 4x - 6y - 3 = 0$

Chord of contact of $(\alpha,\ \beta)$ w.r.t circle is

$\alpha x + \beta y - 2(x + \alpha) - 3(\beta + y) - 3 = 0$

$(\alpha - 2)x + (\beta - 3)y - (2\alpha + 3\beta + 3) = 0$

But equation of chord of contact is $x + y - 3 = 0$

Comparing the coefficients,

$x + y - 3 = 0$

$\frac{\alpha - 2}{1} = \frac{\beta - 3}{1} = -\frac{2\alpha + 3\beta + 3}{-3}$

$\Rightarrow \alpha = -6,\ \beta = -5$

$\therefore 4\alpha - 7\beta = 11$

19. Consider a sequence $a_1,\ a_2,\ \cdots,\ a_n$ given by $a_n = a_{n-1} + 2^{n-1}$, $a_1 = 1$ and another sequence given by $b_n = b_{(n-1)} + a_{n-1}$, $b_1 = 1$. Also $P = \sum_{n=1}^{10} \frac{b_n}{2^n}$ and $Q = \sum_{n=1}^{10} \frac{n}{2^{n-1}}$, then $2^7(P - 2Q)$ is __________.

Answer (7.5)

Solution:

$a_2 - a_1 = 2^1$

$a_3 - a_2 = 2^2$

$\dots$

$a_n - a_{n-1} = 2^{n-1}$

$a_n = 2^n - 1$

$b_2 - b_1 = a_1$

$b_3 - b_2 = a_2$

$\dots$

$b_n - b_{n-1} = a_{n-1}$

$b_n = 2^n - n$

$P - 2Q = \sum_{n=1}^{10} \frac{2^n - n}{2^n} - \frac{2n}{2^{n-1}}$

$= \sum_{n=1}^{10} \left(1 - \frac{5n}{2^n}\right) = 10 - 5\sum_{n=1}^{10} \frac{n}{2^n}$

Let $S_n = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} \quad \cdots(1)$

$\frac{S_n}{2} = \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots + \frac{n}{2^{n+1}} \quad \cdots(2)$

By subtracting eq.(2) from eq.(1) we get,

$\frac{S_n}{2} = \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\right) - \frac{n}{2^{n+1}}$

$\frac{S_n}{2} = \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\right) - \frac{n}{2^{n+1}}$

$\Rightarrow \frac{S_n}{2} = \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{\frac{1}{2}} - \frac{n}{2^{n+1}}$

$\Rightarrow S_n = 2\left(1 - \left(\frac{1}{2}\right)^n - \frac{n}{2^{n+1}}\right)$

$\Rightarrow S_{10} = 2\left(1 - \left(\frac{1}{2}\right)^{10} - \frac{10}{2^{11}}\right)$

$= 2\left(1 - \frac{12}{2^{11}}\right)$

$P - 2Q = 10 - 5 \times 2\left(1 - \frac{12}{2^{11}}\right)$

$P - 2Q = 10 - 10 + \frac{120}{2^{11}} = \frac{60}{2^{10}}$

$\therefore 2^7(P - 2Q) = 7.50$